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UNIVERSITY OF WISCONSIN-MADISON

A SEQUENTIAL PROCEDURE FOR DETERMINING
THE LENGTH OF A
STEADY-STATE SIMULATION

by
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A Sequential Procedure for Determining the Length of a Steady-State Simulation*

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A common problem faced by simulators is that of constructing a confidence interval for the steady-state mean of a stochastic process. We have reviewed the existing procedures for this problem and found that they all produce confidence intervals with coverages which may be considerably lower than desired. Thus, in many cases simulators will have more confidence in their results than is justified.

In this paper we present a new sequential procedure based on the method of batch means for constructing a confidence interval with coverage close to the desired level. Empirical results for a large number of stochastic systems indicate that the new procedure performs quite well.

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Let $\{X_i, i \geq 1\}$ be a stochastic process with steady-state mean

$$\mu = \lim_{n \to \infty} \sum_{i=1}^{n} X_i/n \text{ (with probability 1).}$$

A common problem faced by simulators is that of constructing a confidence interval (c.i.) for μ. There are two basic approaches: (1) Construct a c.i. from an arbitrary fixed sample size (2) Sequentially increase the sample size until an "acceptable" c.i. can be constructed.

For the fixed sample size approach, five methods have been suggested in the simulation literature: replication, batch means, spectrum analysis, autoregressive representation, and regeneration cycles. (See Crane and Iglehart [6,7], Fishman [10,11], Iglehart [12], and Law [14].) Unfortunately, all of these methods have the drawback that if the total sample size is chosen too small, then the actual coverage of a constructed c.i. may be considerably lower than desired. This was empirically shown for replication and batch means in [14], and will be demonstrated for the other three methods in Law [16]. (These results were reported at the 1976 Winter Simulation Conference.)

If the total sample size is made sufficiently large, then the actual coverage will be close to the desired level; however, what is sufficiently large for one stochastic system may not be adequate for another. Thus, in practice a simulator will not know the actual coverage of his c.i.

The above summary suggests that a procedure is needed to determine the total sample size necessary to achieve "acceptable" coverage. Three such procedures have been suggested. Fishman [9] applied a procedure based on autoregressive representation to 100 independent simulation runs of an $M/M/1$ queue with $\rho = .9$. On each run he attempted to construct a 90% c.i. for the mean number of customers in system, L. He found that between 66 and 79 percent of the c.i.'s covered L, depending on the choice of the initial sample size. Robinson [20] applied a similar sequential procedure based on regeneration cycles to 100 runs of the $M/M/1$ queue with $\rho = .5$. On each run he attempted to construct a 90% c.i. for the mean delay in queue, d. He found that between 60 and 63 percent of the c.i.'s covered d, depending on the initial sample size. Mechanic and McKay [17] developed a procedure based on batch means that was difficult to understand and was never substantially tested. However, it was their work that provided the initial motivation for the sequential procedure presented in the next section.

The remainder of this paper is organized as follows. In Sections 1 and 2, respectively, we describe and justify the new procedure. The results of testing the procedure on a large number of queueing, inventory, and computer models are given in Section 3. Finally, in Section 4 we summarize our findings and comment on the computational efficiency of the procedure.

1. The Procedure

Suppose we make a simulation run of length n and then divide the resulting observations $X_1, X_2, \ldots, X_n$ into k batches of length m $(n=k \cdot m)$. Let $\overline{X}_j(m)$ $(j=1,2,\ldots,k)$ be the sample mean of the m observations in the

jth batch and let $\bar{\bar{X}}(k,m) = \sum_{j=1}^{k} \bar{X}_j(m)/k$ be the grand sample mean. If m is sufficiently large, then the $\bar{X}_j(m)$'s will be essentially uncorrelated (see Section 2) and an approximate $100(1 - \alpha)\%$ c.i. for μ is given by

$$\bar{\bar{X}}(k,m) \pm t_{k-1,1-\alpha/2}\sqrt{\hat{\sigma}^2[\bar{\bar{X}}(k,m)]}\ , \tag{1}$$

where $t_{k-1,1-\alpha/2}$ is the $1-\alpha/2$ point for a t distribution with k-1 degrees of freedom and

$$\hat{\sigma}^2\left[\bar{\bar{X}}(k,m)\right] = \sum_{j=1}^{k}\left[\bar{X}_j(m) - \bar{\bar{X}}(k,m)\right]^2 \Big/ k(k-1). \tag{2}$$

This approach to constructing a c.i. is called the method of batch means (see [14]).

The validity of the c.i. given by (1) *depends crucially on the* $\bar{X}_j(m)$'s being approximately uncorrelated (see Section 2). We will attempt to determine the presence of significant correlation by estimating $\rho_1(m)$, the lag 1 correlation between the $\bar{X}_j(m)$'s. The usual estimator of $\rho_1(m)$ is

$$\hat{\rho}_1(k,m) = \sum_{j=1}^{k-1}\left[\bar{X}_j(m)-\bar{\bar{X}}(k,m)\right]\left[\bar{X}_{j+1}(m)-\bar{\bar{X}}(k,m)\right]\Big/\sum_{j=1}^{k}\left[\bar{X}_j(m)-\bar{\bar{X}}(k,m)\right]^2 .$$

However, if $\hat{\rho}_1^1(k/2,m)$ and $\hat{\rho}_1^2(k/2,m)$ are, respectively, the usual lag 1 estimators based on the first $k/2$ and last $k/2$ batches (k is assumed to be even), then we can also estimate $\rho_1(m)$ by the jackknifed estimator

$$\tilde{\rho}_1(k,m) = 2\hat{\rho}_1(k,m) - [\hat{\rho}_1^1(k/2,m) + \hat{\rho}_1^2(k/2,m)]/2.$$

We will use $\tilde{\rho}_1(k,m)$ rather than $\hat{\rho}_1(k,m)$ to estimate $\rho_1(m)$ since, in general, it will be less biased (see Miller [18]).

We now state our sequential procedure.

Step 0. Let $c = .225$, $n_0 = 600$, $n_1 = 800$, and $i = 1$.

Step 1.a. Divide the n_i observations into 400 batches of size $m = n_i/400$. Compute $\tilde{\rho}_1(400,m)$ from $\overline{X}_j(m)$ ($j=1,2,\ldots,400$). If $\tilde{\rho}_1(400,m) \geq c$, go to Step 1.c. If $\tilde{\rho}_1(400,m) \leq 0$ (see Note 1), go to Step 2. Otherwise, go to Step 1.b.

b. Divide n_i into 200 batches of size $2m$. Compute $\tilde{\rho}_1(200,2m)$ from $\overline{X}_j(2m)$ ($j=1,2,\ldots,200$). (See Note 2.) If $\tilde{\rho}_1(200,2m) < \tilde{\rho}_1(400,m)$ (see Note 3), go to Step 2. Otherwise, go to Step 1.c.

c. Replace i by $i + 1$, set $n_i = 2n_{i-2}$ (see Note 4), collect the additional observations required, and go to Step 1.a.

Step 2. Divide n_i into 40 batches of size $10m$. Use $\overline{X}_j(10m)$ ($j=1,2,\ldots,40$) (see Note 2) in (1) to construct a c.i. for μ.

Notes:

1. If $\rho_1(m) \leq 0$, then, as is discussed in Section 2, 40 batches of size $10m$ will most likely produce a c.i. with at least $100(1 - \alpha)\%$ coverage.

2. An appropriate number of the $\overline{X}_j(m)$'s may be averaged to compute the $\overline{X}_j(2m)$'s or the $\overline{X}_j(10m)$'s.
3. If $0 < \rho_1(m) < .225$ and if $\rho_1(m')$ is decreasing for $m' \geq m$, then batches of size $10m$ are approximately uncorrelated (see Section 2). We test to see if $\rho_1(m')$ is decreasing for $m' \geq m$ by checking to see whether $\rho_1(2m) < \rho_1(m)$.
4. The successive sample sizes considered are 800,1200,1600,2400,... . Thus, the total sample size is doubled every other iteration.

The sequential procedure which is described above uses 400 batches of size m to decide when 40 batches of size $10m$ are uncorrelated. By using 400 rather than 40 batches, the correlation estimator, which is used to determine the stopping point, has a smaller bias and variance. In fact, the procedure will not work well if the correlation is estimated directly from 40 batches.

The next section gives a justification of the procedure. The reader who is primarily interested in the procedure's performance may proceed directly to Section 3.

2. Justification

In the following two subsections we discuss the general form of the sequential procedure and the choice of the stopping value c.

A. General Form of the Procedure

Suppose that the observations $X_1, X_2, \ldots, X_n$ are from a covariance stationary process. For $i=0,1,\ldots,n-1$, let $C_i = Cov[X_j, X_{j+i}]$ and for $i=0,1,\ldots,k-1$, let $C_i(m) = Cov[\overline{X}_j(m), \overline{X}_{j+i}(m)]$ and $\rho_i(m) = C_i(m)/C_0(m)$. The following lemma, which is proved in the appendix, shows that

$\rho_i(m) \simeq 0$ $(i \geq 1)$ for sufficiently large m.

Lemma 1. If $0 < \sum_{\ell=-\infty}^{\infty} C_\ell < \infty$, then $\rho_i(m) \to 0$ as $m \to \infty$ for $i=1,2,\ldots,k-1$.

Let $b(k,m)$ be defined by

$$E\{\hat{\sigma}^2[\bar{\bar{X}}(k,m)]\} = b(k,m)\sigma^2[\bar{\bar{X}}(k,m)],$$

where $\hat{\sigma}^2[\bar{\bar{X}}(k,m)]$ was given by (2). We now prove that $\hat{\sigma}^2[\bar{\bar{X}}(k,m)]$ is asymptotically unbiased as $m \to \infty$.

Theorem 2. If $0 < \sum_{\ell=-\infty}^{\infty} C_\ell < \infty$, then $b(k,m) \to 1$ as $m \to \infty$ for all $k \geq 2$.

Proof: It is easy to show that (see [14])

$$b(k,m) = \frac{\left\{k\bigg/\left[1 + 2\sum_{i=1}^{k-1}(1 - i/k)\rho_i(m)\right]\right\} - 1}{k - 1}.$$

The desired result follows since $\rho_i(m) \to 0$ as $m \to \infty$ for $i=1,2,\ldots,k-1$ by the lemma.

The above expression for $b(k,m)$ shows that $\hat{\sigma}^2[\bar{\bar{X}}(k,m)]$ is unbiased when $\rho_i(m) = 0$ for $i=1,2,\ldots,k-1$, has a negative bias $(b(k,m) < 1)$ when $\rho_i(m) > 0\ \forall i$, and has a positive bias $(b(k,m) > 1)$ when $\rho_i(m) < 0\ \forall i$. The case $\rho_i(m) > 0\ \forall i$ is of the greatest concern since $\sigma^2[\bar{\bar{X}}(k,m)]$ will then be underestimated and the coverage of the resulting c.i. is likely to be less than desired.

If we choose m large enough so that the $\overline{X}_j(m)$'s are approximately normally distributed in addition to being uncorrelated, then it becomes plausible to proceed as if the $\overline{X}_j(m)$'s were independent identically distributed normal random variables (r.v.'s) and to use (1) to construct a c.i. for μ.

There are three potential sources of error when using (1) to construct a c.i.:

(1) Bias in $\hat{\sigma}^2[\overline{\overline{X}}(k,m)]$ when m is too small for the $\overline{X}_j(m)$'s to be uncorrelated.

(2) Nonnormality of the $\overline{X}_j(m)$'s.

(3) The fact that $\{X_i, i \geq 1\}$ is not, in practice, covariance stationary.

However, for simple queueing models (e.g., $M/M/1$) Law [14] found that the bias in $\hat{\sigma}^2[\overline{\overline{X}}(k,m)]$ was the most serious source of error and that nonnormality was not a problem for k approximately 20 or more. This suggests that a sequential procedure based on batch means must be able to determine that batch size, m, for which the $\overline{X}_j(m)$'s are approximately uncorrelated.

To determine the types of correlation which can occur in practice, we studied the following processes for which $\rho_i(m)$ and $b(k,m)$ can be analytically computed:

(1) $\{D_i, i \geq 1\}$ for the $M/M/1$ queue (see Daley [8]) with ρ = .5,.8, and .9, where D_i is the delay in queue of the ith customer.

(2) $\{E_i, i \geq 1\}$ for an (s,S) inventory system (see [14) for details), where E_i is the expenditure in the ith period.

(3) Thirty different AR(1), AR(2), and ARMA(1,1) time series models (see Box and Jenkins [3, p. 46]) with parameters chosen over the entire range of feasible values.

From studying these 34 stochastic processes, we found essentially three types of behavior for $\rho_1(m)$ as a function of m, examples of which are shown in Figures 1, 2, and 3. For type 1 behavior the lag 1 correlation $\rho_1(m)$ strictly decreases to zero. If for some m, $\rho_1(m) < .4$, then $.9 < b(40,10m) < 1$ and $\rho_1(10m) \approx .05$. That is, if $\rho_1(m) < .4$, then the variance estimator based on 40 batches of size $10m$ is approximately unbiased. The $M/M/1$ queue exhibits type 1 behavior.

In type 2 behavior, $\rho_1(m)$ changes direction one or more times and then strictly decreases to zero. If for some m, $\rho_1(m) < .4$ and $\rho_1(m')$ is decreasing for $m' \geq m$, then $.9 < b(40,10m) < 1$ and $\rho_1(10m) \approx .05$. The $M/M/1$ queue with service in random order (SIRO) is of this type (see Figure 4).

For type 3 behavior, $\rho_1(m) < 0$ and $b(40,10m) > 1$, for all m. In this case the $\overline{X}_j(10m)$'s may be correlated, but the coverage will be at least as great as that desired. The (s,S) inventory system exhibits type 3 behavior.

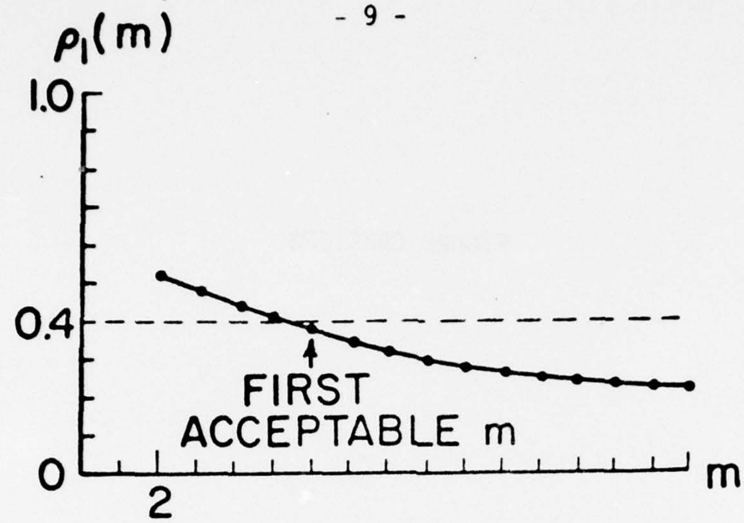
We certainly do not claim that the above three types of behavior are the only ones that can occur. In fact, for some of the time series models we studied, $\rho_1(m)$ can be positive or negative. However, if $\rho_1(m) > 0$ for some m, then we found that type 1 behavior is followed, and if $\rho_1(m) < 0$ for some m, then behavior similar to that of type 3 is followed.

FIGURE CAPTIONS

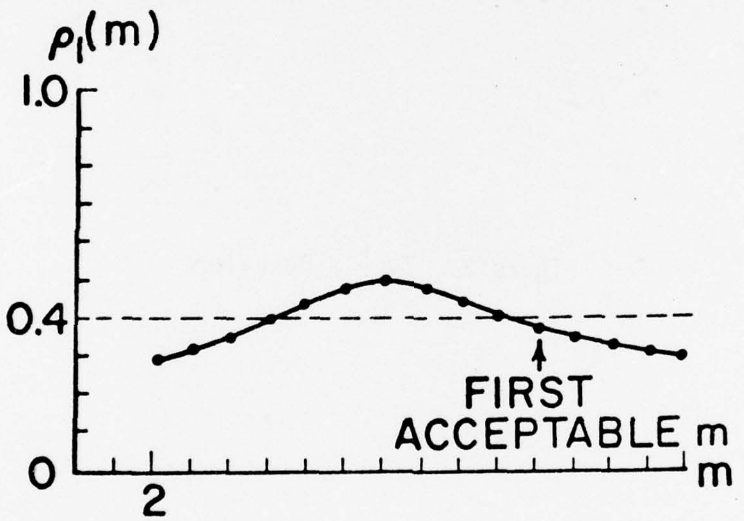
Figure 1. Type 1 Behavior.

Figure 2. Type 2 Behavior.

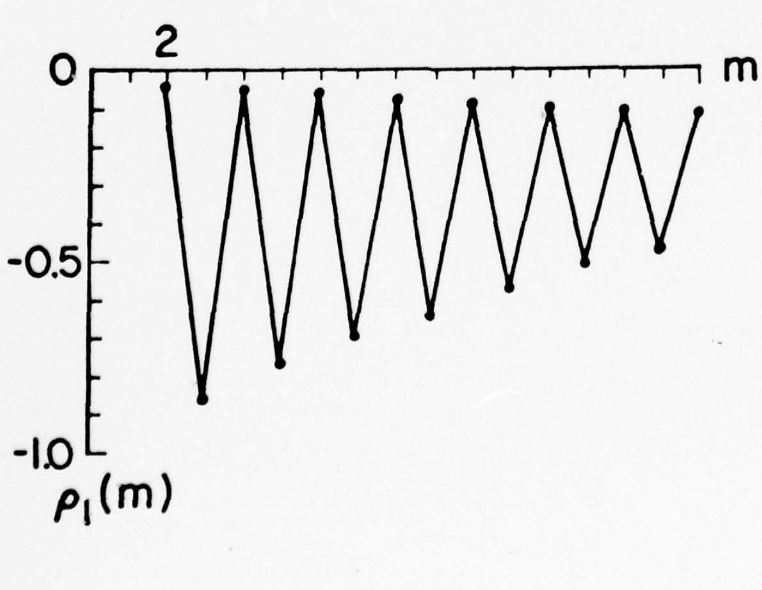
Figure 3. Type 3 Behavior.



$\rho_1(m)$
1.0
0.4
0
2
m
FIRST
ACCEPTABLE m



$\rho_1(m)$
1.0
0.4
0
2
m
FIRST
ACCEPTABLE m



2
0
m
-0.5
-1.0
$\rho_1(m)$

Types 1 and 2 behavior described above explain why the procedure checks for $\rho_1(m) < c$ and $\rho_1(m')$ decreasing for $m' \geq m$. Type 3 behavior explains why it checks for $\rho_1(m) \leq 0$.

B. The Choice of the Stopping Value c

The above discussion was predicated upon knowing $\rho_1(m)$ which, in fact, is estimated by $\tilde{\rho}_1(400,m)$. Thus, it is possible for $\rho_1(m)$ to be much larger than .4, but to have the estimate $\tilde{\rho}_1(400,m) < .4$, which might result in the procedure's stopping prematurely. To determine how small c should be to account for the sampling variability of $\tilde{\rho}_1(400,m)$, we applied our sequential procedure with various values of c to the $M/M/1$ FIFO queue, the $M/M/1$ LIFO queue, and the $M/M/1$ SIRO queue, each with $\rho = .8$. For each system we made 200 independent simulation runs of the process $\{D_i, i \geq 1\}$ and attempted to construct 90% c.i.'s for $d = 3.2$. The results of these simulations are given in Table I. Note that for the FIFO and SIRO disciplines, $c = .375$ gives good coverage, but that for the LIFO case a much smaller value of c is required. We chose $c = .225$ as our stopping value because we felt that a true coverage of 80% is the minimum acceptable coverage for a "90% c.i." A smaller value of c would result in better coverage for LIFO, but also in a considerable increase in sample size for the other models. (These additional data would not be wasted if a smaller c.i. half length was desired.)

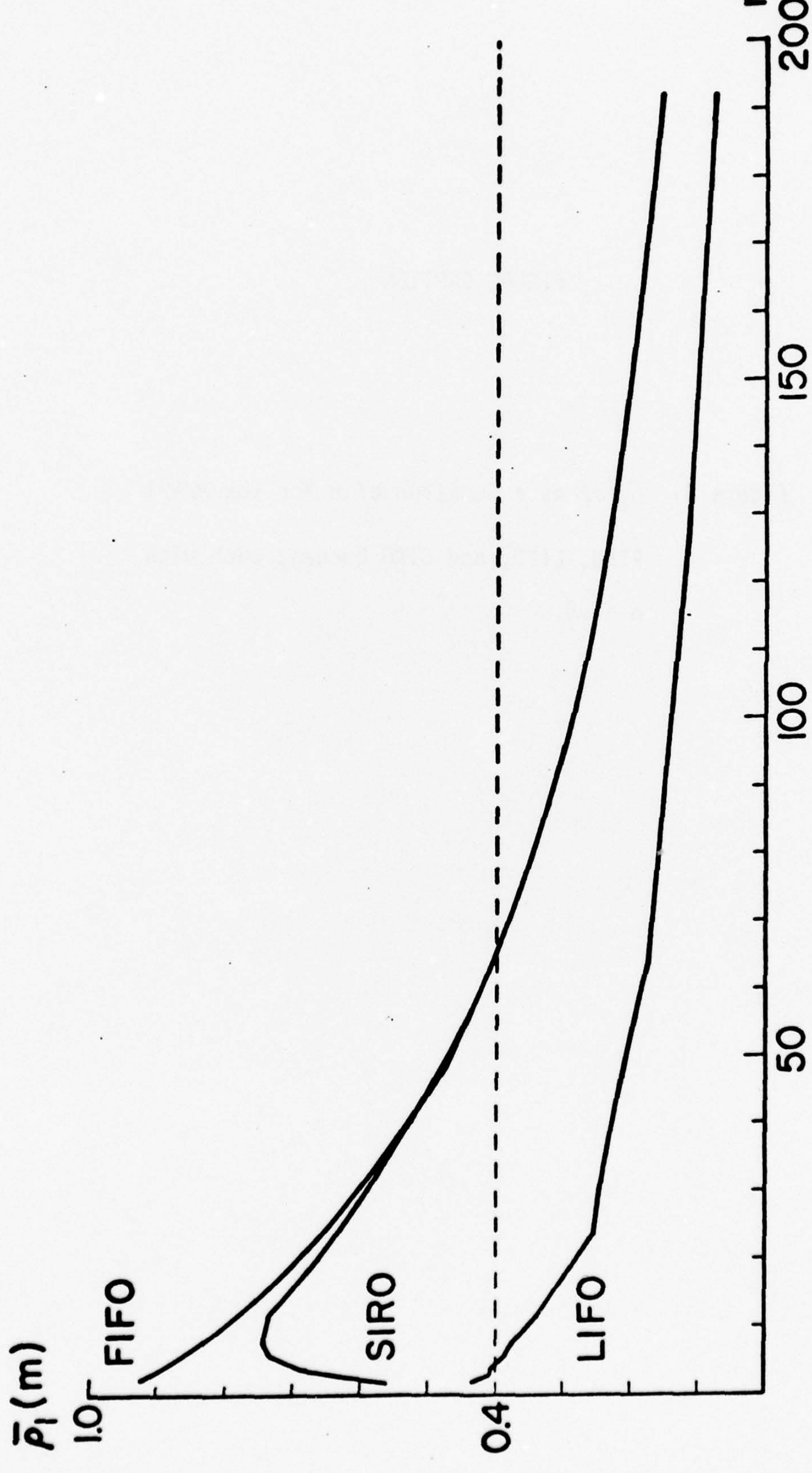
The fact that a smaller value of c is required for the LIFO model is due not to the variability of $\tilde{\rho}_1(400,m)$, but rather to $\rho_1(m)$'s being a more slowly decreasing function of m than for any of the other 44 models considered in this paper. This can be seen in Figure 4 where we plot $\overline{\rho}_1(m)$ (the average of the 200 values of $\tilde{\rho}_1(400,m)$) as a function

TABLE I. Proportion of Coverage and Average Sample Size for Various Values of c.

	M/M/1 FIFO $\rho = .8$		M/M/1 LIFO $\rho = .8$		M/M/1 SIRO $\rho = .8$	
c	Coverage	Average Sample Size	Coverage	Average Sample Size	Coverage	Average Sample Size
.375	.900	36464	.650	3882	.880	35616
.250	.900	59776	.760	15054	.860	59680
.225	.890	67584	.805	19100	.865	69280
.200	.890	77952	.825	26796	.860	77472
.150	.910	100992	.840	41058	.865	99008

FIGURE CAPTION

Figure 4. $\overline{\rho}_1(m)$ as a Function of m for the $M/M/1$ FIFO, LIFO, and SIRO Queues, each with $\rho = .8$.


$\bar{P}_1(m)$
1.0
0.4
FIFO
SIRO
LIFO
50
100
150
200
m

of m for the three queues. Observe that LIFO starts lower than FIFO and SIRO but decreases more slowly. In particular, $\overline{\rho}_1(m) < .4$ for $m = 6$, but $\overline{\rho}_1(10m) \approx .19$ ($\overline{\rho}_1(m) < .4$ implies $\overline{\rho}_1(10m) \approx .05$ for the other models). By choosing $c = .225$, we are designing the sequential procedure to perform adequately in the "worst" case.

3. Empirical Results

In order to see how well the sequential procedure works, we simulated a large number of stochastic systems for which analytical results are available. The results of these simulations are presented in the next three subsections.

The random numbers, $\{U_i, i \geq 1\}$, used in these simulations were generated from the following generator which is available on the Univac 1110:

$$Y_i = (5^{15} Y_{i-1} + 1) \bmod 2^{35} \quad (i=1,2,\ldots)$$

$$U_i = Y_i / 2^{35} \quad (i=1,2,\ldots),$$

where Y_0 is a specified seed. For a discussion of this generator, see Coveyou and Macpherson [5].

A. Queueing Systems

We first considered a variety of simple queueing systems. Let E_4, M, and H_2 denote, respectively, the 4-Erlang, the exponential, and the hyperexponential distributions with coefficients of variation .5, 1, and 2. (See Law [13] for further discussion of H_2.) For each of the queueing systems $M/M/1$ ($\rho = .5$), $E_4/M/1$ ($\rho = .8$), $M/H_2/1$ ($\rho = .8$), $M/M/2$ ($\rho = .8$), $M/M/1/M/1$ ($\rho = .8$), we made 100 independent simulation runs of the stochastic process $\{D_i, i \geq 1\}$ and attempted to construct 90% c.i.'s for d. For each system, $E(A) = 1$ (the mean interarrival time) and $D_1 = 0$ (i.e., no customers are present at time zero). In Table II we give for each system d, the proportion of the 100 c.i.'s which covered d, the average sample size at termination, and the average c.i. half length divided by d. (We will henceforth call the average c.i. half length divided by μ the "relative precision" of the c.i.) For completeness we also give the previous results for the $M/M/1$ queue with $\rho = .8$ and FIFO, LIFO, and SIRO queue disciplines. Observe that the average sample size at termination is quite system dependent. For example, the $M/M/1$ queue with $\rho = .8$ requires a sample size about 8 times larger than that required by the $M/M/1$ queue with $\rho = .5$. This is, of course, due to the fact that the process $\{D_i, i \geq 1\}$ is more correlated for larger values of ρ.

For the $M/M/1$ queue it can be shown that (see [8]) an approximate theoretical sample size of

$$n = \frac{[E(A)]^2\rho^3(2 + 5\rho - 4\rho^2 + \rho^3)}{(1 - \rho)^4}\left[\frac{t_{39,.95}}{\delta d}\right]^2 \tag{3}$$

is required to obtain a c.i. whose half length is δd ($0 < \delta < 1$). Since the average half length for $\rho = .5$ was .098 of d (see Table II), if we substitute $\delta = .098$ into (3), we get a theoretical sample size of 8572. Similarly, substitution of $\delta = .068$ into (3) gives a theoretical sample size of 75813 for $\rho = .8$. These sample sizes compare closely to those actually obtained by the procedure.

B. An Inventory System

The second type of stochastic system we considered was an (s,S) inventory system with zero delivery lag and backlogging. This system

TABLE II. Empirical Results for Queueing Systems.

System	ρ	d	Number of Runs	Proportion of Coverage	Average Sample Size	Relative Precision
$M/M/1$ FIFO	.5	.50	100	.850	8352	.098
$M/M/1$ FIFO	.8	3.20	200	.890	67584	.068
$M/M/1$ LIFO	.8	3.20	200	.805	19100	.141
$M/M/1$ SIRO	.8	3.20	200	.865	69280	.067
$E_4/M/1$	.8	1.81	100	.830	42240	.075
$M/H_2/1$	.8	8.00	100	.800	188928	.072
$M/M/2$	.8	2.84	100	.910	65792	.076
$M/M/1/M/1$	.8	6.40	100	.900	88832	.049

is described in detail in [14]. We made 100 independent simulation runs of the process $\{E_i, i \geq 1\}$ and attempted to construct 90% c.i.'s for the steady-state mean expenditure per period, e = 112.108. We found that each run terminated on the first iteration (n = 800) and that 99 out of 100 c.i.'s covered e. These results can be explained by the fact that $\rho_i(m) < 0$ $(i \geq 1)$ and $b(40,10m) > 1$ for this system (see [14] for some actual values).

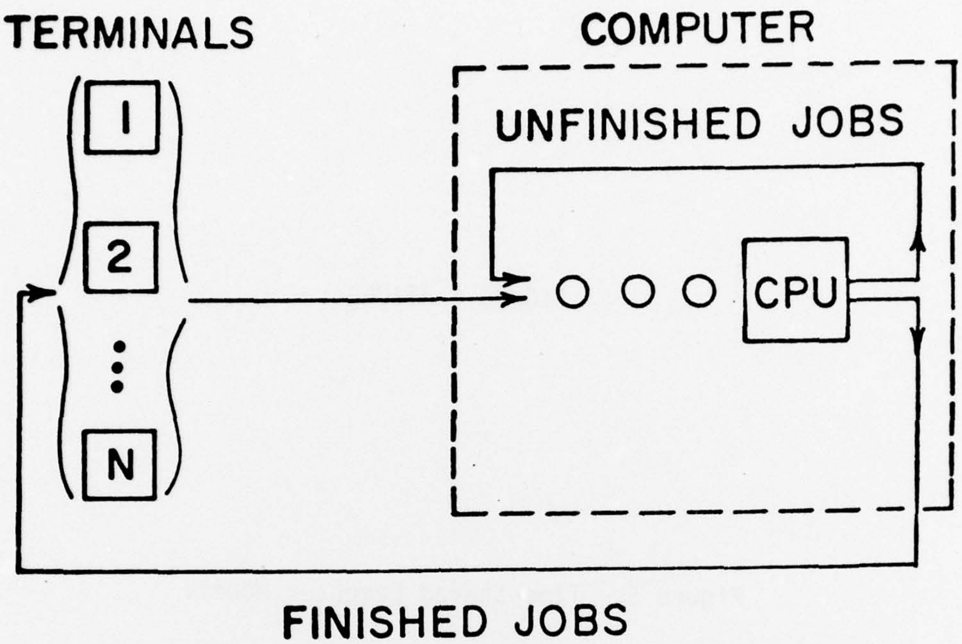
C. Computer Models

To see how the sequential procedure works on more complex systems, we simulated two models of computer systems. These models resemble more closely the type of system that might actually be simulated in the real world than do the simple models of the previous two subsections. We first considered a model of a time-shared computer system which was studied by Adiri and Avi-Itzhak [1] and more recently in a simulation context by Sargent [19]. There are N terminals and a single central processing unit (CPU) as shown in Figure 5. The operator of each terminal "thinks" for an amount of time which is an exponential r.v. with rate μ_1 and then sends a message with service time which is an exponential r.v. with rate μ_2. The arriving jobs join a FIFO queue at the CPU. The CPU allocates to each job a maximum service quantum of length q. If the (remaining) service time of a job, s, is less than or equal to q, then the CPU spends time s plus τ (a fixed overhead) processing the job and the job returns to its terminal. If $s > q$, then the CPU spends time q plus τ processing the job and the job joins the end of the queue. This process is repeated until the job's service time is eventually completed.

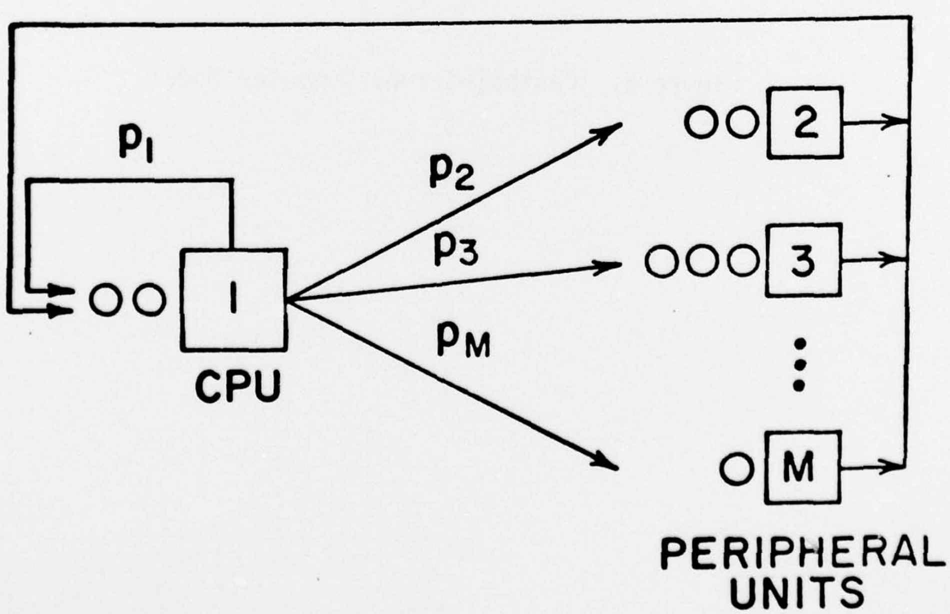
FIGURE CAPTIONS

Figure 5. Time-Shared Computer Model.

Figure 6. Central-Server Computer Model.



TERMINALS
COMPUTER
UNFINISHED JOBS
1
2
N
CPU
FINISHED JOBS



p_1
p_2
p_3
p_M
1
CPU
2
3
M
PERIPHERAL UNITS

We let $N = 35$, $\mu_1 = 1/25$, $\mu_2 = 5/4$, $q = .1$, $\tau = .015$, and then simulated the stochastic process $\{R_i, i \geq 1\}$, where R_i is the response time of the ith job requesting service and all terminals are in the think state at time zero. Our objective was to construct a 90% c.i. for the steady-state mean response time, $r = 8.246$. We made 100 independent runs and obtained a coverage of .92, an average sample size of 36160, and a relative precision of .036.

We next considered the central-server model of a multiprogrammed computer system (see Buzen [4]). There is a CPU (unit 1) and M-1 peripheral units (units 2 through M) as shown in Figure 6. Each unit has its own FIFO queue and the service time at unit i is an exponential r.v. with rate μ_i (i=1,2,...,M). It is assumed that there are N jobs in the system at all times. When a job completes service at the CPU it leaves the system with probability p_1 or it goes to peripheral unit i with probability p_i (i=2,3,...,M), a job leaving the system is instantaneously replaced by a new job which joins the end of the CPU queue. A job leaving a peripheral unit also joins the CPU queue.

We made 100 simulation runs of the process $\{R_i, i \geq 1\}$ for four different sets of parameters and on each run we attempted to construct a 90% c.i. for r (the steady-state mean time between entries to the CPU queue). The parameters for these models and the results of the simulations are given in Table III. Also included are the steady-state probability that unit i is busy, ρ_i, and the state of the system at time zero, $(\ell_1, \ell_2, \ldots, \ell_M)$, where ℓ_i is the number of jobs at unit i.

Table III. Empirical Results for the Central-Server Computer Model with $M = 3$, $\mu_1 = 1$, and $p_1 = 0$.

N	μ_2	μ_3	p_2	p_3	ρ_1	ρ_2	ρ_3	r	Initial State	Proportion of Coverage	Average Sample Size	Relative Precision
4	.50	.50	.5	.5	.67	.67	.67	6.000	(1,1,2)	.84	1548	.038
8	.50	.50	.5	.5	.80	.80	.80	10.000	(1,1,6)	.92	3396	.027
8	.45	.05	.9	.1	.44	.88	.88	18.279	(5,1,2)	.85	1588	.096
8	1.79	.20	.9	.1	.99	.50	.50	8.072	(7,1,0)	.89	2976	.032

4. Summary

We have used the sequential procedure to construct "90% c.i.'s" for 13 stochastic systems (excluding the inventory model), obtaining coverages between .80 and .92 and relative precisions between .027 and .141. Averaging over the 13 systems, we obtain a mean coverage of .867 and a mean relative precision of .067.

The sequential procedure described here is fairly easy to program, is computationally efficient, and requires only 800 storage locations to store the batch means. A FORTRAN program for the procedure and an explanation of how to use it may be found in Law and Carson [15].

Appendix

Our objective is to prove Lemma 1 (see Section 2).

Lemma A.1. If $\sum_{\ell=1}^{\infty} C_\ell$ converges, then $\sum_{\ell=1}^{m-1} \ell C_\ell / m \to 0$ as $m \to \infty$.

Proof: $$\sum_{\ell=1}^{m-1} \ell C_\ell / m = \sum_{\ell=1}^{m-1} C_\ell - \sum_{\ell=1}^{m-1} (m - \ell) C_\ell / m.$$

The result follows since the second sum on the right-hand side converges to $\sum_{\ell=1}^{\infty} C_\ell$ by Lemma 8.3.1 of Anderson [2, p. 460].

Lemma A.2. If $\sum_{\ell=1}^{\infty} C_\ell$ converges, then $S_i^1(m) \equiv \sum_{j=1}^{m-1} (1 - j/m) C_{im+j} \to 0$ as $m \to \infty$ for $i \geq 1$.

Proof: $$S_i^1(m) = \sum_{\ell=im+1}^{(i+1)m-1} C_\ell - \sum_{\ell=1}^{(i+1)m-1} \ell C_\ell/m + \sum_{\ell=1}^{im} \ell C_\ell/m + i \sum_{\ell=im+1}^{(i+1)m-1} C_\ell.$$

The first and fourth sums converge to 0 since $\sum_{\ell=1}^{\infty} C_\ell$ converges. The second and third sums converge to 0 by Lemma A.1.

Lemma A.3. If $\sum_{\ell=1}^{\infty} C_\ell$ converges, then $S_i^2(m) \equiv \sum_{j=1}^{m-1} (1-j/m)C_{im-j} \to 0$ as $m \to \infty$ for $i \geq 1$.

Proof: $$S_i^2(m) = \sum_{j=1}^{m-1} (im-j)C_{im-j}/m - (i-1)\sum_{j=1}^{m-1} C_{im-j}$$

$$= \sum_{\ell=1}^{im-1} \ell C_\ell/m - \sum_{\ell=1}^{(i-1)m} \ell C_\ell/m - (i-1) \sum_{\ell=(i-1)m+1}^{im-1} C_\ell.$$

The first and second sums converge to 0 by Lemma A.1. The third sum converges to 0 since $\sum_{\ell=1}^{\infty} C_\ell$ converges.

Lemma 1. If $0 < \sum_{\ell=-\infty}^{\infty} C_\ell < \infty$, then $\rho_i(m) \to 0$ as $m \to \infty$ for $i=1,2,\ldots,k-1$.

Proof: Since $\rho_i(m) = mC_i(m)/mC_0(m)$ and $mC_0(m) \to \sum_{\ell=-\infty}^{\infty} C_\ell$ as $m \to \infty$

(see [2, p. 459]), it suffices to show that $mC_i(m) \to 0$ as $m \to \infty$. However, $mC_i(m) = C_{im} + S_i^1(m) + S_i^2(m)$ (see [14]) and $C_{im} \to 0$ as $m \to \infty$ since $\sum_{\ell=1}^{\infty} C_\ell$ converges. The desired result now follows from Lemmas A.2 and A.3.

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A common problem faced by simulators is that of constructing a confidence interval for the steady-state mean of a stochastic process. We have reviewed the existing procedures for this problem and found that they all produce confidence intervals with coverages which may be considerably lower than desired. Thus, in many cases simulators will have more confidence in their results than is justified. → next page

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In this paper we present a new sequential procedure based on the method of batch means for constructing a confidence interval with coverage close to the desired level. Empirical results for a large number of stochastic systems indicate that the new procedure performs quite well.